Computation of Elastic Constants of Functionally Graded Materials Using Eigenfunction Virtual Fields Method

Nigamaa Nayakanti and S. J. Subramanian

Department of Engineering Design
Indian Institute of Technology Madras

ASME 2013 International Mechanical Engineering Congress & Exposition
Table of contents

1 Introduction
   - Virtual Fields Method - VFM
   - Eigenfunction Virtual Fields Method - EVFM
Table of contents

1 Introduction
   - Virtual Fields Method - VFM
   - Eigenfunction Virtual Fields Method - EVFM

2 Application of EVFM to Functionally Graded Materials
Table of contents

1 Introduction
   - Virtual Fields Method - VFM
   - Eigenfunction Virtual Fields Method - EVFM

2 Application of EVFM to Functionally Graded Materials

3 Summary
Outline

1 Introduction
   - Virtual Fields Method - VFM
   - Eigenfunction Virtual Fields Method - EVFM

2 Application of EVFM to Functionally Graded Materials

3 Summary
Introduction

Virtual Fields Method - VFM

VFM is a popular technique for estimating constitutive properties from full-field data

- Pioneered by Prof. Grediac, Prof. Pierron and co-workers\(^1\)
- Inverse-computation technique for material property estimation
- Linear and nonlinear material models (e.g. linear elasticity, elasto-plasticity, visco-elasticity, visco-plasticity)
- Anisotropy, heterogeneity

---

\(^1\) F. Pierron and M. Grediac, 2012. *The Virtual Fields Method*, Springer
VFM is based on the Principle of Virtual Work

\[ \int_V \sigma : \epsilon^* dV = \int_{S_T} t \cdot u^* dA \]

- \( V \) → volume occupied by the solid of interest
- \( S_T \) → portion of external surface over which tractions are specified
- \( \sigma \) → statically admissible stress field
- \( u^* \) → kinematically admissible virtual displacement field
- \( \epsilon^* \) → virtual strain field
- \( t \) → true tractions applied over \( S_T \)
We deal with infinitesimal deformation of a linear elastic solid in this work.

\[ \sigma = C \epsilon \]

Voigt notation:
\[
(\sigma_{11}, \sigma_{22}, \sigma_{12}) \equiv (\sigma_1, \sigma_2, \sigma_3); (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12}) \equiv (\epsilon_1, \epsilon_2, \epsilon_3)
\]

Principle of Virtual Work
\[ C_{ij} \int_V \epsilon_i \epsilon_j^* dV = \int_S t_i u_i^* dA \]

Yields one equation for a given virtual field
Choosing adequate number of VFs allows us to solve for the unknown material properties

1. Choose a virtual displacement (strain) field
2. Differentiate (integrate) to get virtual strain (displacement)
3. Substitute in PVW to get one equation in the unknowns
4. Repeat for various VFs to generate system of linear equations

\[ C_{ij} \int_V \epsilon_i \epsilon_j^* dV = \int_{S_T} t_i u_i^* dA \]
Choosing adequate number of VFs allows us to solve for the unknown material properties

1. Choose a virtual displacement (strain) field
2. Differentiate (integrate) to get virtual strain (displacement)
3. Substitute in PVW to get one equation in the unknowns
4. Repeat for various VFs to generate system of linear equations

\[
C_{ij} \int_V \epsilon_i \epsilon_j^* \, dV = \int_{ST} t_i u_i^* \, dA \implies PQ = R
\]
Central to the success of VFM is the choice of virtual fields

- Intuitively chosen in the beginning\(^2\)
- Special virtual fields\(^3\)
- Piecewise virtual fields\(^4\)
- Chosen to minimize sensitivity to noise\(^5\)

In EVFM

VFMs are computed from the measured strain fields


Overview of Eigenfunction Virtual Fields Method

Obtain strain fields

Compute eigenfunctions of strain fields

Identify and construct virtual fields from dominant eigenfunctions

Compute PVW integrals & assemble $f(Q) = 0$

Solve $f(Q) = 0$

Initial guess for $Q$

Tolerance

Accept Solution
Overview of Eigenfunction Virtual Fields Method

- Obtain strain fields
- Compute eigenfunctions of strain fields
- Identify and construct virtual fields from dominant eigenfunctions
- Compute PVW integrals and assemble $f(Q) = 0$
- Solve $f(Q) = 0$
- Initial guess for $Q$
- Tolerance
- Accept solution
Overview of Eigenfunction Virtual Fields Method

- Obtain strain fields
- Compute eigenfunctions of strain fields

Initial guess for $Q$

Tolerance

Accept Solution

Nigamaa, Subramanian
EVFM for FGM
ASME-IMECE 2013
9 / 26
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble $f(Q) = 0$
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble $f(Q) = 0$
5. Solve $f(Q) = 0$
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble \( f(Q) = 0 \)
5. Solve \( f(Q) = 0 \)
6. Initial guess for \( Q \)
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble $f(Q) = 0$
5. Solve $f(Q) = 0$
6. Initial guess for $Q$
7. Tolerance
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble $f(Q) = 0$
5. Solve $f(Q) = 0$
6. Accept Solution

Initial guess for $Q$

Tolerance
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble \( f(Q) = 0 \)
5. Solve \( f(Q) = 0 \)
6. Accept Solution
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble $f(Q) = 0$
5. Solve $f(Q) = 0$
6. Accept Solution
Principal Component Analysis (PCA) is used to analyze measured strain fields.\(^6\)

- Spatial patterns (eigenfunctions) along rows and columns are identified through PCA

Principal Component Analysis (PCA) is used to analyze measured strain fields.\(^6\)

- Spatial patterns (eigenfunctions) along rows and columns are identified through PCA.
- Only a fraction of eigenfunctions are important.

---

Principal Component Analysis (PCA) is used to analyze measured strain fields

- Spatial patterns (eigenfunctions) along rows and columns are identified through PCA
- Only a fraction of eigenfunctions are important
- By choosing only dominant eigenfunctions

---

Principal Component Analysis (PCA) is used to analyze measured strain fields\(^6\).

- Spatial patterns (eigenfunctions) along rows and columns are identified through PCA.
- Only a fraction of eigenfunctions are important.
- By choosing only dominant eigenfunctions:
  - most of the data is retained
  - most of the noise is eliminated
  - dimensionality is reduced

---

PCA of strain fields is easily performed through Singular Value Decomposition (SVD)

SVD: \( E_{(m \times n)} = L_{(m \times m)} S_{(m \times n)} R^T_{(n \times n)} \)
PCA of strain fields is easily performed through Singular Value Decomposition (SVD)

\[
E_{(m \times n)} = L_{(m \times m)} S_{(m \times n)} R^T_{(n \times n)}
\]

- \( L = [l_1 \ l_2 \ l_3 \ \ldots \ l_m] \) contains left singular vectors
PCA of strain fields is easily performed through Singular Value Decomposition (SVD)

\[
\text{SVD : } \mathbf{E}_{(m \times n)} = \mathbf{L}_{(m \times m)} \mathbf{S}_{(m \times n)} \mathbf{R}^T_{(n \times n)}
\]

- \(\mathbf{L} = [l_1 \quad l_2 \quad l_3 \ldots \quad l_m]\) contains left singular vectors
- \(\mathbf{R} = [r_1 \quad r_2 \quad r_3 \ldots \quad r_m]\) contains right singular vectors

\(\mathbf{S}\) contains \(\alpha = \text{rank}(\mathbf{E})\) non-zero singular values \(\lambda_i, i = 1, \alpha\) along the diagonal, all other elements are zero
PCA of strain fields is easily performed through Singular Value Decomposition (SVD):

$$\text{SVD} : \quad E_{(m \times n)} = L_{(m \times m)} S_{(m \times n)} R^T_{(n \times n)}$$

- $L = [l_1 \quad l_2 \quad l_3 \quad ... \quad l_m]$ contains left singular vectors
- $R = [r_1 \quad r_2 \quad r_3 \quad ... \quad r_m]$ contains right singular vectors
- singular vectors are eigenfunctions
PCA of strain fields is easily performed through Singular Value Decomposition (SVD)

\[
E_{(m\times n)} = L_{(m\times m)} S_{(m\times n)} R^T_{(n\times n)}
\]

- \(L = [l_1 \ l_2 \ l_3 \ \ldots \ l_m]\) contains left singular vectors
- \(R = [r_1 \ r_2 \ r_3 \ \ldots \ r_m]\) contains right singular vectors
- Singular vectors are eigenfunctions
- \(S\) contains \(\alpha = \text{rank}(E)\) non-zero singular values \(\lambda_i, \ i = 1, \alpha\) along the diagonal, all other elements are zero
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble \( f(Q) = 0 \)
5. Solve \( f(Q) = 0 \)
6. Initial guess for \( Q \)
7. Tolerance
8. Accept Solution
By choosing only dominant eigenfunctions, dimensionality and noise are reduced

- $p$ dominant eigenfunctions are chosen for strain reconstruction

\[
E = \sum_{i=1}^{\alpha} \lambda_i l_i(\mathbf{r}_i)^T \approx \sum_{i=1}^{p} \lambda_i l_i(\mathbf{r}_i)^T
\]

In EVFM

VSFs are piecewise continuous functions of dominant eigenfunctions
Overview of Eigenfunction Virtual Fields Method

1. Obtain strain fields
2. Compute eigenfunctions of strain fields
3. Identify and construct virtual fields from dominant eigenfunctions
4. Compute PVW integrals & assemble $f(Q) = 0$
5. Solve $f(Q) = 0$
6. Initial guess for $Q$
7. Tolerance
8. Accept Solution

Nigamaa, Subramanian
EVFM for FGM
ASME-IMECE 2013
Each row and column are expressed using eigenfunctions and their principal components.

\[ k^{th} \text{ row:} \]

\[
E_k^* = \sum_{t=1}^{p} \left( E_k^* \cdot r_t \right) \{ r_t \}^T = \sum_{t=1}^{p} \left( \lambda_{tk} \right) \{ r_t \}^T
\]

\[ \lambda_{tk} = \psi_k \to i^{th} \text{ principal component of } k^{th} \text{ row} \]

\[ k^{th} \text{ column:} \]

\[
E^*_k = \sum_{t=1}^{p} \left( E^*_k \cdot l_t \right) \{ l_t \} = \sum_{t=1}^{p} \left( \lambda_{rk} \right) \{ l_t \} \]

\[ \lambda_{rk} = \xi_k \to i^{th} \text{ principal component of } k^{th} \text{ column} \]
Each row and column are expressed using eigenfunctions and their principal components

$k^{th}$ row:

$$E_{k}^{*} = \sum_{t=1}^{p} (E_{k}^{*} \cdot r_{t}) \{r_{t}\}^{T} = \sum_{t=1}^{p} (\lambda_{t} l_{t}^{k}) \{r_{t}\}^{T}$$

$$\lambda_{t} l_{t}^{k} = \psi_{t}^{k} \rightarrow i^{th} \text{ principal component of } k^{th} \text{ row}$$
Each row and column are expressed using eigenfunctions and their principal components

\( k^{th} \) row:

\[
E^{k*} = \sum_{t=1}^{p} (E^{k*} \cdot r_t) \{r_t\}^T = \sum_{t=1}^{p} (\lambda_t l_t^k) \{r_t\}^T
\]

\( \lambda_t l_t^k = \psi_t^k \rightarrow i^{th} \) principal component of \( k^{th} \) row

\( k^{th} \) column:

\[
E^{*k} = \sum_{t=1}^{p} \{l_t\} (E^{*k} \cdot l_t) = \sum_{t=1}^{p} \{l_t\} (\lambda_t r_t^k)
\]

\( \lambda_t r_t^k = \xi_t^k \rightarrow i^{th} \) principal component of \( k^{th} \) column
Orthogonality of eigenfunctions leads to simplification of equations

PVW equation involves integrals of type $\int \varepsilon_i \varepsilon_j^* dV$
Orthogonality of eigenfunctions leads to simplification of equations

The PVW equation involves integrals of type \( \int_V \varepsilon_i \varepsilon_j^* dV \)

\[
\int_V \varepsilon_i \varepsilon_j^* dV = \sum_{k=1}^{n} \int_{V_k} \varepsilon_i \varepsilon_j^* dV
\]

- \( \varepsilon_j \) is a continuous function of \( p \) dominant left eigenfunctions
- \( \varepsilon_j^* \) is a continuous function of \( s^{th} \) dominant left eigenfunction

\[
\int \varepsilon_i \varepsilon_j^* dV = \sum_{k=1}^{n} \int_{V_k} \varepsilon_i \varepsilon_j^* dV = h\Delta X_1 \Delta X_2 \sum_{k=1}^{n} \xi_k
\]

\[
\int_{S_T} t_i u_j^* dS \text{ is simplified using integrals of chosen virtual strain field}
\]
Outline

1. Introduction
   - Virtual Fields Method - VFM
   - Eigenfunction Virtual Fields Method - EVFM

2. Application of EVFM to Functionally Graded Materials

3. Summary
EVFM is used to estimate elastic properties of a Functionally Graded Material

Exponential variation of Young’s modulus and constant Poisson’s ratio are assumed.

$$E = E_0 e^{(X_1/\beta)}$$

$E_0, \beta, \nu$ are to be computed from strain data.

Square plate with hole made of such material is subjected to uniaxial tension test.
EVFM is used to estimate elastic properties of a Functionally Graded Material

- Exponential variation of Young’s modulus and constant Poisson’s ratio are assumed. $E = E_0 e^{(x_1/\beta)}$
EVFM is used to estimate elastic properties of a Functionally Graded Material

- Exponential variation of Young’s modulus and constant Poisson’s ratio are assumed. $E = E_0 e^{(X_1/\beta)}$
- $E_0, \beta, \nu$ are to be computed from strain data
EVFM is used to estimate elastic properties of a Functionally Graded Material

- Exponential variation of Young’s modulus and constant Poisson’s ratio are assumed. \( E = E_0 e^{(X_1/\beta)} \)
- \( E_0, \beta, \nu \) are to be computed from strain data
- Square plate with hole made of such material is subjected to uniaxial tension test
Virtual Fields are generated from eigenfunctions of measured strain fields

\[ i\varepsilon_1^* = 0; \quad i\varepsilon_2^* = l_i; \quad \varepsilon_6^* = 0 \]

\[ iu_2^* = 0; \quad iu_2^* = \int_0^{x_2} \varepsilon_2^* dX \]
Virtual Fields are generated from eigenfunctions of measured strain fields

\[ i \varepsilon^*_1 = 0; \quad i \varepsilon^*_2 = l_i; \quad \varepsilon^*_6 = 0 \]
\[ i u^*_2 = 0; \quad i u^*_2 = \int_0^{x_2} \varepsilon^*_2 dX \]

\[ \int V (Q_2 \varepsilon_1 + Q_1 \varepsilon_2) e^{(x_1/\beta)} i \varepsilon^*_2 dV = \int_{S_T} t_2 i u^*_2 dS \]
\[ Q_1 = \frac{E_0}{(1 - \nu^2)}; \quad Q_2 = \frac{E_0 \nu}{(1 - \nu^2)} \]
Virtual Fields are generated from eigenfunctions of measured strain fields

\[
\begin{bmatrix}
\int_V \varepsilon_2 e^{(X_1/\beta)} \varepsilon_2^* dV \\
\int_V \varepsilon_1 e^{(X_1/\beta)} \varepsilon_2^* dV \\
\int_V \varepsilon_2 e^{(X_1/\beta)} \varepsilon_2^* dV \\
\int_V \varepsilon_1 e^{(X_1/\beta)} \varepsilon_2^* dV
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2
\end{bmatrix}
= 
\begin{bmatrix}
\int_{S_T} t_2 \varepsilon_2^* dS 
\end{bmatrix}
\]
Virtual Fields are generated from eigenfunctions of measured strain fields

Virtual Field 2

\[
\begin{pmatrix}
\int_V \varepsilon_2 e^{(x_1/\beta)} \varepsilon_2^* dV & \int_V \varepsilon_1 e^{(x_1/\beta)} \varepsilon_2^* dV \\
\int_V \varepsilon_2 e^{(x_1/\beta)} \varepsilon_2^* dV & \int_V \varepsilon_1 e^{(x_1/\beta)} \varepsilon_2^* dV
\end{pmatrix}
\begin{pmatrix}
Q_1 \\
Q_2
\end{pmatrix}
= 
\begin{pmatrix}
\int_{S_T} t_2^1 u_2^* dS \\
\int_{S_T} t_2^2 u_2^* dS
\end{pmatrix}
\]
Virtual Fields are generated from eigenfunctions of measured strain fields

Virtual Field 3

\[
\begin{bmatrix}
\int_V \varepsilon_2 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_1 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_2 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_1 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_2 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_1 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_2 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\int_V \varepsilon_1 e^{(X_1/\beta)} \varepsilon^*_2 dV \\
\end{bmatrix}
\begin{bmatrix}
Q_1 \\
Q_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\int_{S_T} t_2 \varepsilon^*_2 dS \\
\end{bmatrix}
\]
A system of non-linear EVFM equations are obtained

- System of equations $PQ = R$ takes the form

$$P_{i1}(\beta)Q_1 + P_{i2}(\beta)Q_2 = R_i \quad (i = 1, 2, 3)$$
A system of non-linear EVFM equations are obtained

- System of equations \( \mathbf{PQ} = \mathbf{R} \) takes the form

\[
P_{i1}(\beta)Q_1 + P_{i2}(\beta)Q_2 = R_i \quad (i = 1, 2, 3)
\]

\( P_{i1} \) and \( P_{i2} \) contain \( \beta \) which is to be evaluated
A system of non-linear EVFM equations are obtained

- System of equations \( PQ = R \) takes the form
  \[
P_{i1}(\beta)Q_1 + P_{i2}(\beta)Q_2 = R_i \quad (i = 1, 2, 3)
  \]

  \( P_{i1} \) and \( P_{i2} \) contain \( \beta \) which is to be evaluated

- Solved in an iterative manner minimizing a cost function \( C \) defined as
A system of non-linear EVFM equations are obtained

- System of equations $\mathbf{PQ} = \mathbf{R}$ takes the form
  
  $$P_{i1}(\beta)Q_1 + P_{i2}(\beta)Q_2 = R_i \quad (i = 1, 2, 3)$$

  $P_{i1}$ and $P_{i2}$ contain $\beta$ which is to be evaluated

- Solved in an iterative manner minimizing a cost function $C$ defined as

  Cost-function: $C = \|\mathbf{R} - \mathbf{PQ}\|_2$
Good results are obtained for noise-free data

Parameter | $E_0$ (GPa) | $\nu$ | $\beta$ (mm) | $\gamma$
---|---|---|---|---
Values input | 10.00 | 0.300 | 6.213
Values obtained | 9.96 | 0.299 | 6.207

Global Minimum at $1/\beta = 0.2$
Good results are obtained for noise-free data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$E_0$(GPa)</th>
<th>$\nu$</th>
<th>$\beta$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values input</td>
<td>10.00</td>
<td>0.300</td>
<td>6.213</td>
</tr>
<tr>
<td>Values obtained</td>
<td>9.96</td>
<td>0.299</td>
<td>6.207</td>
</tr>
</tbody>
</table>
Outline

1. Introduction
   - Virtual Fields Method - VFM
   - Eigenfunction Virtual Fields Method - EVFM

2. Application of EVFM to Functionally Graded Materials

3. Summary
Conclusions

- In EVFM
  - Eigenfunctions of measured strain fields are used as VFs
  - Orthogonality of eigenfunctions leads to enormous simplification of the computations
- EVFM has been applied to a FGM
  - Recovered parameters are in good agreement with the true values
Conclusions

- In EVFM
  - Eigenfunctions of measured strain fields are used as VFs
  - Orthogonality of eigenfunctions leads to enormous simplification of the computations

- EVFM has been applied to a FGM
  - Recovered parameters are in good agreement with the true values

Outlook

- Effect of noise in measured strains on computed parameters
- Study of material inhomogeneity at a much smaller scale
- Dealing with missing data and regions with discontinuities
Thank You!